

# Online Square-into-Square Packing

Sándor P. Fekete · Hella-Franziska Hoffmann

**Abstract** In 1967, Moon and Moser proved a tight bound on the critical density of squares in squares: any set of squares with a total area of at most  $1/2$  can be packed into a unit square, which is tight. The proof requires full knowledge of the set, as the algorithmic solution consists in sorting the objects by decreasing size, and packing them greedily into shelves. Since then, the online version of the problem has remained open; the best upper bound is still  $1/2$ , while the currently best lower bound is  $1/3$ , due to Han et al. (2008). In this paper, we present a new lower bound of  $11/32$ , based on a dynamic shelf allocation scheme, which may be interesting in itself.

We also give results for the closely related problem in which the size of the square container is not fixed, but must be dynamically increased in order to accommodate online sequences of objects. For this variant, we establish an upper bound of  $3/7$  for the critical density, and a lower bound of  $1/8$ . When aiming for accommodating an online sequence of squares, this corresponds to a  $2.82\dots$ -competitive method for minimizing the required container size, and a lower bound of  $1.33\dots$  for the achievable factor.

**Keywords** Packing · online problems · packing squares · critical density.

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A preliminary extended abstract appears in the Proceedings of the 16th International Workshop APPROX 2013 [2].

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## 1 Introduction

Packing is one of the most natural and common optimization problems. Given a set  $\mathcal{O}$  of objects and a container  $E$ , find a placement of all objects into  $E$ , such that no two overlap. Packing problems are highly relevant in many practical applications, both in geometric and abstract settings. Simple one-dimensional variants (such as the PARTITION case with two containers, or the KNAPSACK problem of a largest packable subset) are NP-hard. Additional difficulties occur in higher dimensions: as Leung et al. [10] showed, it is NP-hard even to check whether a given set of squares fits into a unit-square container.

When dealing with an important, but difficult optimization problem, it is crucial to develop a wide array of efficient methods for distinguishing feasible instances from the infeasible ones. In one dimension, a trivial necessary and sufficient criterion is the total size of the objects in comparison to the container. This makes it natural to consider a similar approach for the two-dimensional version: *What is the largest number  $\delta$ , such that any family of squares with area at most  $\delta$  can be packed into a unit square?* An upper bound of  $\delta \leq 1/2$  is trivial: two squares of size  $1/2 + \varepsilon$  cannot be packed. As Moon and Moser showed in 1967 [12],  $\delta = 1/2$  is the correct critical bound: sort the objects by decreasing size, and greedily pack them into a vertical stack of one-dimensional “shelves”, i.e., horizontal subpackings whose height is defined by the largest object.

This approach cannot be used when the set of objects is not known a priori, i.e., in an online setting. It is not hard to see that a pure shelf-packing approach can be arbitrarily bad. However, other, more sophisticated approaches were able to prove lower bounds for  $\delta$ : the current best bound (established by Han et al. [5]) is based on a relatively natural recursive approach and shows that  $\delta \geq 1/3$ .

Furthermore, it may not always be desirable (or possible) to assume a fixed container: the total area of objects may remain small, so a fixed large, square container may be wasteful. Thus, it is logical to consider the size of the container itself as an optimization parameter. Moreover, considering a possibly *larger* container reflects the natural optimization scenario in which the full set of objects *must* be accommodated, possibly by paying a price in the container size. From this perspective,  $1/\sqrt{\delta}$  yields a competitive factor for the minimum size of the container, which is maintained at any stage of the process. This perspective has been studied extensively for the case of an infinite strip, but not for an adjustable square.

### 1.1 Our Results

We establish a new best lower bound of  $\delta \geq 11/32$  for packing an online sequence of squares into a fixed square container, breaking through the threshold of  $1/3$  that is natural for simple recursive approaches based on brick-like structures. Our result is based on a two-dimensional system of multi-directional shelves and buffers, which are dynamically allocated and updated. We believe that this approach is interesting in itself, as it may not only yield worst-case estimates, but also provide a possible avenue for further improvements, and be useful as an algorithmic method.

As a second set of results, we establish the first upper and lower bounds for a square container, which is dynamically enlarged, but must maintain its quadratic shape. In particular, we show that there is an upper bound of  $\delta \leq 3/7 < 1/2$  for

the critical density, and a lower bound of  $1/8 \leq \delta$ ; when focusing on the minimum size of a square container, these results correspond to a  $2.82\dots$ -competitive factor, and a lower bound of  $1.33\dots$  for the achievable factor by any deterministic online algorithm.

## 1.2 Related Work

*Offline Packing of Squares.* One of the earliest considered packing variants is the problem of finding a dense square packing for a rectangular container. In 1966 Moser [13] first stated the question as follows:

“What is the smallest number  $A$  such that any family of objects with total area at most 1 can be packed into a rectangle of area  $A$ ?”

The offline case has been widely studied since 1966; there is a long list of results for packing squares into a rectangle. Already in 1967, Moon and Moser [12] gave the first bounds for  $A$ : any set of squares with total area at most 1 can be packed into a square with side lengths  $\sqrt{2}$ , which shows  $A \leq 2$ , and thus  $\delta \geq 1/2$ ; they also proved  $A \geq 1.2$ . Meir and Moser [11] showed that any family of squares each with side lengths  $\leq x$  and total area  $A$  can be packed into a rectangle of width  $w$  and height  $h$ , if  $w, h \geq x$  and  $x^2 + (w-x)(h-x) \geq A$ ; they also proved that any family of  $k$ -dimensional cubes with side lengths  $\leq x$  and total volume  $V$  can be packed into a rectangular parallelepiped with edge lengths  $a_1, \dots, a_k$  if  $a_i \geq x$  for  $i = 1, \dots, k$  and  $x^k + \prod_{i=1}^k (a_i - x) \geq V$ . Kleitman and Krieger improved the upper bound on  $A$  to  $\sqrt{3} \approx 1.733$  [8] and to  $4/\sqrt{6} \approx 1.633$  [9] by showing that any finite family of squares with total area 1 can be packed into a rectangle of size  $\sqrt{2} \times 2/\sqrt{3}$ . Novotný further improved the bounds to  $1.244 \approx (2 + \sqrt{3})/3 \leq A < 1.53$  in 1995 [14] and 1996 [15]. The current best known upper bound of 1.3999 is due to Hougardy [6].

*Online Packing of Squares.* In 1997, Januszewski and Lassak [7] studied the online version of the dense packing problem. In particular, they proved that for  $d \geq 5$ , every online sequence of  $d$ -dimensional cubes of total volume  $2(\frac{1}{2})^d$  can be packed into the unit cube. For lower dimensions, they established online methods for packing (hyper-) cubes and squares with a total volume of at most  $\frac{3}{2}(\frac{1}{2})^d$  and  $\frac{5}{16}$  for  $d \in \{3, 4\}$  and  $d = 2$ , respectively. The results are achieved by performing an online algorithm that subsequently divides the unit square into rectangles with aspect ratio  $\sqrt{2}$ . In the following, we call these rectangles *bricks*. The best known lower bound of  $2(\frac{1}{2})^d$  for any  $d \geq 1$  was presented by Meir and Moser [11].

Using a variant of the brick algorithm, Han et al. [5] extended the result to packing a 2-dimensional sequence with total area  $\leq 1/3$  into the unit square.

A different kind of online square packing was considered by Fekete et al. [3, 4]. The container is an unbounded strip, into which objects enter from above in a Tetris-like fashion; any new object must come to rest on a previously placed object, and the path to its final destination must be collision-free. Their best competitive factor is  $34/13 \approx 2.6154\dots$ , which corresponds to an (asymptotic) packing density of  $13/34 \approx 0.38\dots$

## 2 Packing into a Fixed Container

As noted in the introduction, it is relatively easy to achieve a dense packing of squares in an offline setting: sorting the items by decreasing size makes sure that a shelf-packing approach places squares of similar size together, so the loss of density remains relatively small. This line of attack is not available in an online setting; indeed, it is not hard to see that a brute-force shelf-packing method can be arbitrarily bad if the sequence of items consists of a limited number of medium-sized squares, followed by a large number of small ones. Allocating different size classes to different horizontal shelves is not a remedy, as we may end up saving space for squares that never appear, and run out of space for smaller squares in the process; on the other hand, fragmenting the space for large squares by placing small ones into it may be fatal when a large one does appear after all.

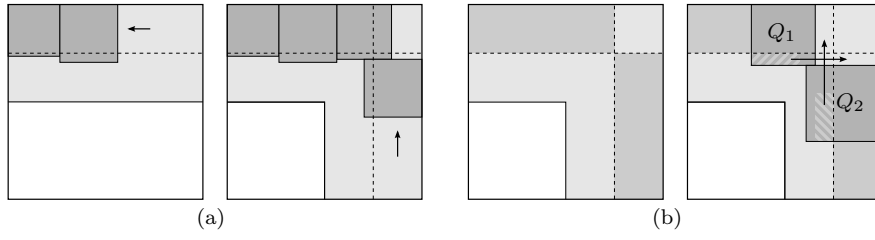
Previous approaches (in particular, the brick-packing algorithm) have side-stepped these difficulties by using a recursive subdivision scheme. While this leads to relatively good performance guarantees (such as the previous record of  $1/3$  for a competitive ratio), it seems impossible to tighten the lower bound; in particular,  $1/3$  seems to be a natural upper bound for this relatively direct approach. Thus, making progress on this natural and classical algorithmic problem requires less elegant, but more powerful tools.

In the following we present a different approach for overcoming the crucial impediment of mixed square sizes, and breaking through the barrier of  $1/3$ . Our *Recursive Shelf Algorithm* aims at subdividing the set of squares into different size classes called *large*, *medium* and *small*, which are packed into pre-reserved shelves. The crucial challenge is to dynamically update regions when one of them gets filled up before the other ones do; in particular, we have to protect against the arrival of one large square, several medium-sized squares, or many small ones. To this end, we combine a number of new techniques:

- Initially, we assign carefully chosen horizontal strips for shelf-packing each size class.
- We provide rules for dynamically updating shelf space when required by the sequence of items. In particular, we accommodate a larger set of smaller squares by inserting additional *vertical* shelves into the space for larger squares whenever necessary.
- In order to achieve the desired overall density, we maintain a set of buffers for overflowing strips. These buffers can be used for different size classes, depending on the sequence of squares.

With the help of these techniques, and a careful analysis, we are able to establish  $\delta \geq 11/32$ . It should be noted that the development of this new technique may be more significant than the numerical improvement of the density bound: we are convinced that tightening the remaining gap towards the elusive  $1/2$  will be possible by an extended (but more complicated) case analysis.

The remainder of this section is organized as follows. In Section 2.1 we give an overview of the algorithm. Section 2.2 sketches the placement of large objects, while Section 2.3 describes the packing created with medium-sized squares. In Section 2.4 we describe the general concept of shelf-packing that is used for the packing of



**Fig. 1** Packing medium squares (Subsection 2.3). (a): The L-shaped packing created with medium squares. (b) Density consideration: The Ceiling Packing Algorithm packs at least as much as the gray area shown on the left. The parts of the packed squares that are used to fill the gaps appearing in the top right corner are depicted as hatched regions.

small squares discussed in Section 2.5. The overall performance is analyzed in Section 2.6.

## 2.1 Algorithm Overview

We construct a shelf-based packing in the unit square by packing *small*, *medium* and *large squares* separately. We stop the Recursive Shelf Algorithm when the packings of two different subalgorithms would overlap. As it turns out, this can only happen when the total area of the given squares is greater than  $11/32$ ; details are provided in the “Combined Analysis” of Section 2.6, after describing the approach for individual size classes.

In the following, we will subdivide the set of possible squares into subsets, according to their size: We let  $H_k$  denote the height class belonging to the interval  $(2^{-(k+1)}, 2^{-k}]$ . In particular, we call all squares in  $H_0$  *large*, all squares in  $H_1$  *medium*, and all other squares (in  $H_{\geq 2}$ ) *small*.

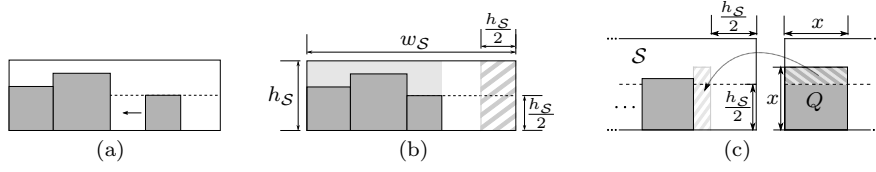
## 2.2 Packing Large Squares

The simplest packing subroutine is applied to large squares, i.e., of size greater than  $1/2$ . We pack a square  $Q_0 \in H_0$  into the top right corner of the unit square  $U$ . Clearly, only one large square can be part of a sequence with total area  $\leq 11/32$ . Hence, this single location for the squares in  $H_0$  is sufficient.

## 2.3 Packing Medium Squares

We pack all medium squares (those with side lengths in  $(1/4, 1/2]$ ) separately; note that there can be at most five of these squares, otherwise their total area is already bigger than  $3/8 > 11/32$ . Moreover, if there is a large square, there can be at most one medium square (otherwise the total area exceeds  $3/8$ ), and both can be packed next to each other.

We start with packing the  $H_1$ -squares from left to right coinciding with the top of the unit square  $U$ . If a square would cross the right boundary of  $U$ , we



**Fig. 2** (a) A shelf packed with squares of one height class. (b) Different areas of a shelf  $S$ .  $occupied(S)$ : total area of squares (dark gray),  $usedSection(S)$ : region with light gray background (incl.  $occupied(S)$ ) and  $end(S)$ : hatched region to the right. (c) Assignment of  $extra(Q)$  (hatched) to  $S$  when square  $Q$  causes an overflow of shelf  $S$ .

continue by placing the following squares from top to bottom coinciding with the right boundary; see Fig. 1(a).

We call the corresponding subroutine the *Ceiling Packing Algorithm*. Without interference of other height classes, the algorithm succeeds in packing any sequence of  $H_1$ -squares with total area  $\leq 3/8$ .

**Theorem 1** *The Ceiling Packing Subroutine packs any sequence of medium squares with total area at most  $3/8$  into the unit square.*

*Proof.* The Ceiling Packing subroutine successfully packs every incoming square until a square  $Q$  causes an overflow of the second shelf  $S_2$  in the bottom right corner of  $U$ . We prove, that the total area of the given sequence  $\sigma$  must be greater than the gray region  $R$  depicted in Fig. 1(b). The area  $\|R\|$  of  $R$  equals  $2 \cdot \frac{1}{4} \cdot \frac{3}{4} = 3/8$ . Note that the remaining free space in the top right corner might be greater than  $1/4 \cdot 1/4$ . However, if that is the case, then the two packed squares ( $Q_1, Q_2$ ) defining the top right hole must be large. Hence, there must be a part of each of the two squares that extends over the respective half of its corresponding packing shelf. Similar to the situation at the end of a usual shelf, we can use the extra space of  $Q_1$  and  $Q_2$  to fill the gaps appearing in the top right corner; see Fig. 1(b). Overall, we have  $\|\mathcal{P}\| + \|Q\| > \|R\| = 2 \cdot \frac{1}{4} \cdot \frac{3}{4} = 3/8$ .  $\square$

## 2.4 Shelf Packing

In this section we use the well-known shelf-packing algorithm that is used for packing small squares into the unit square. Given a set of squares with maximum size  $h$ , a *shelf*  $S$  is a subrectangle of the container that has height  $h$ ; a *shelf packing* places the squares into a shelf next to each other, until some object no longer fits; see Fig. 2(a). When that happens, the shelf is closed, and a new shelf gets opened. Before we analyze the density of the resulting packing, we introduce some notation.

*Notation.* In the following we call a shelf with height  $2^{-k}$  designed to accommodate squares of height class  $H_k$  an  $H_k$ -shelf. We let  $w_S$  denote the width of a shelf  $S$ ,  $h_S$  denote its height and  $\mathcal{P}(S)$  denote the set of squares packed into it. We define  $usedSection(S)$  as the horizontal section of  $S$  that contains  $\mathcal{P}(S)$ ; see Fig. 2(b). We denote the last  $h_S$ -wide section at the end of  $S$  by  $head(S)$  and the last  $h_S/2$ -wide slice by  $end(S)$ . The total area of the squares packed into a shelf  $S$  is  $occupied(S)$ . The part of the square  $Q$  extending over the upper half of  $S$  is  $extra(Q)$ .

A useful property of the shelf-packing algorithm is that  $usedSection(\mathcal{S})$  has a packing-density of  $1/2$  if we pack  $\mathcal{S}$  with squares of the same height class only. The gap remaining at the end of a closed shelf may vary depending on the sequence of squares. However, the following density property described in the following lemma (due to Moon and Moser [12]).

**Lemma 1** *Let  $\mathcal{S}$  be an  $H_k$ -shelf with width  $w_{\mathcal{S}}$  and height  $h_{\mathcal{S}}$  that is packed with a set  $\mathcal{P}$  of squares all belonging to  $H_k$ . Let  $Q$  be an additional square of  $H_k$  with side length  $x$  that does not fit into  $\mathcal{S}$ . Then the total area  $\|\mathcal{P}\|$  of all the squares packed into  $\mathcal{S}$  plus the area  $\|Q\|$  of  $Q$  is greater than  $\|\mathcal{S}\|/2 - (h_{\mathcal{S}}/2)^2 + \frac{1}{2}h_{\mathcal{S}} \cdot x$ .*

In other words: If we count the extra area of the overflowing square  $Q$  towards the density of a closed shelf  $\mathcal{S}$ , we can, w.l.o.g., assume that  $\mathcal{S}$  has a packing density of  $1/2$ , except for at its end  $end(\mathcal{S})$ . We formalize this charging scheme as follows. When a square  $Q$  causes a shelf  $\mathcal{S}$  to be closed, we assign  $extra(Q)$  to  $\mathcal{S}$ ; see Fig. 2(c). The total area assigned this way is referred to as  $assigned(\mathcal{S})$ . We let  $\tilde{\mathcal{A}}(\mathcal{S})$  denote the total of occupied and assigned area of  $\mathcal{S}$  minus the extra area of the squares in  $\mathcal{P}(\mathcal{S})$ .

**Corollary 1** *Let  $\mathcal{S}$  be a closed shelf packed by the shelf-packing algorithm. Then  $\tilde{\mathcal{A}}(\mathcal{S}) \geq \|\mathcal{S} \setminus end(\mathcal{S})\|/2$ .*

## 2.5 The *packSmall* Subroutine

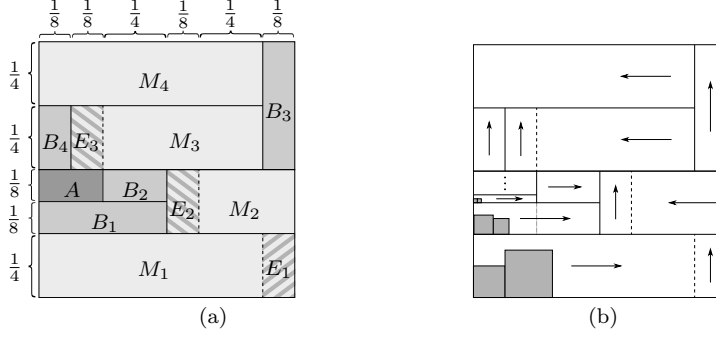
As noted above, the presence of one large or few medium squares already assigns a majority of the required area, without causing too much fragmentation. Thus, the critical question is to deal with small squares in a way that leaves space for larger ones, but allows us to find extra space for a continuing sequence of small squares.

We describe an algorithm for packing any family of small squares with total area up to  $11/32$  in Section 2.5.1 and discuss the density of the resulting packing in Section 2.5.2.

### 2.5.1 Algorithm

In the Recursive Shelf Algorithm we pack all small squares according to the *packSmall* subroutine, independent of the large and medium square packings. The method is based on the general shelf-packing scheme described above.

*Notation and Distribution of the Shelves.* The general partition of the unit square we use is depicted in Fig. 3(a). The regions  $M_1, \dots, M_4$  (in that order) act as shelves for height class  $H_2$ . We call the union  $M$  of the  $M_i$  the *main packing area*; this is the part of  $U$  that will definitely be packed with squares by our *packSmall* subroutine. The other regions may stay empty, depending on the sequence of incoming small squares. The regions  $B_1, \dots, B_4$  provide shelves for  $H_3$ . We call the union  $B$  of the  $B_j$  the *buffer region*. In the region  $A$  we reserve  $H_k$ -shelf space for every  $k \geq 4$ . We call  $A$  the *initial buffer region*. The ends  $E_1$ ,  $E_2$  and  $E_3$  of the main packing regions  $M_1$ ,  $M_2$  and  $M_3$  serve as both: parts of the main packing region and additional buffer areas. We call the union  $E$  of these  $E_i$  the *end buffer region*.



**Fig. 3** (a) Distribution of the shelves for the *smallPack* Algorithm. (b) Initial shelf packing and packing directions.

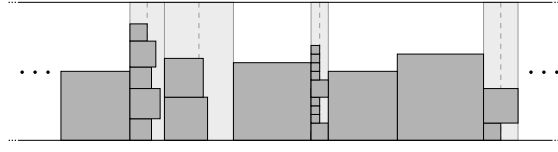
*Packing Approach.* During the packing process, we maintain open shelves for all the height classes for which we already received at least one square as input; we pack each of them according to the shelf-packing scheme described above. We first give a brief overview of how we allocate the respective shelves.

- We start packing small squares into shelves that we open on the left side of the lower half  $\mathcal{H}_\ell$  of  $U$ ; see Fig. 3(b). The region  $M_1$  serves as the first  $H_2$ -shelf, the left half (width  $1/4$ ) of  $B_1$  serves as the first shelf for  $H_3$  and region  $A$  is reserved for first shelves for any  $H_k$  with  $k \geq 4$ ; see details below.
- Once an overflow occurs in a main packing region  $M_i$ , we close the corresponding  $H_2$ -shelf and continue packing  $H_2$ -squares into  $M_{i+1}$ .
- Once the packing for  $H_k$  with  $k \geq 3$  in the initial  $H_k$ -shelf  $\mathcal{I}$  would exceed a length of  $1/4$ , we close  $\mathcal{I}$ , cut a vertical slice  $\mathcal{V}$  out of the currently active  $H_2$ -shelf (one of the  $M_i$  regions) and treat  $\mathcal{V}$  as the currently active  $H_k$ -shelf.
- If the currently active  $H_k$ -shelf  $\mathcal{V}$  is a vertical shelf and the next incoming  $H_k$ -square  $Q$  would intersect with  $\text{head}(\mathcal{V})$  for the first time, then we allocate space in the buffer region  $B \cup E$  to pack  $Q$  (and potentially other  $H_k$ -squares); see details below. After the buffer packing is performed we go back to using the vertical shelf  $\mathcal{V}$  for  $H_k$ -packing.
- If an overflow occurs in a vertical  $H_k$ -shelf  $\mathcal{V}$ , then we close  $\mathcal{V}$ , cut another vertical slice out of the main packing region and treat it as the next active  $H_k$ -shelf.

We claim that we can accommodate any family of small squares with total area up to  $11/32$  this way. In the following, we describe the packings for the different small height classes in more detail.

*Packing  $H_2$ -squares.* In the main packing area, we always maintain an open shelf for height class  $H_2$ , which is packed with  $H_2$ -squares, as described in Section 2.4. In order to avoid early collisions with large and medium squares, we start with packing  $M_1$  from left to right, continuing with packing  $M_2$  from right to left. Then we alternately treat  $M_3$  and  $M_4$  as the current main packing region, placing  $H_2$ -squares into the region whose *usedSection* is smaller. When the length of *usedSection*( $M_4$ ) becomes larger than  $3/8$ , we prefer  $M_3$  over  $M_4$  until  $M_3$  is full. This way, we can use the end  $E_3$  as an additional buffer for vertical shelves in





**Fig. 4** Sample packing of square into a part of the main packing region according to the vertical shelf packing scheme.

$M_4$  while ensuring that no overlap with the packing of medium squares can occur, unless the total area of the input exceeds  $11/32$ .

*Initial Packing of  $H_{\geq 3}$ -squares.* For  $H_{k \geq 4}$  we reserve space in the initial buffer area  $A$ . As soon as we receive the first square of a height class  $H_k$  with  $k \geq 4$ , we open an  $H_k$ -shelf of length  $1/4$  (and height  $2^{-k}$ ) on top of the existing shelves in  $A$ . We call this shelf *initialBuffer(k)* and pack it from left to right with all subsequent  $H_k$ -squares according to the shelf-packing concept. If *initialBuffer(k)* is full, we start allocating space in form of *vertical  $H_k$ -shelves* in the main packing region.

The same initial buffer packing is performed with  $H_3$ -squares, with the difference that we use  $B$  instead of  $A$  as the initial buffer packing location. As soon as the placement of a square  $Q$  would cause the initial buffer packing for  $H_3$  in  $B$  to exceed a total length of  $1/4$ , we open a vertical  $H_3$ -shelf with  $Q$  in the main packing region and stop using  $B$  for the initial buffer packing of  $H_3$ .

*Vertical Shelves.* In contrast to the situation in general shelf packings, we do not only pack  $H_2$ -squares into the  $H_2$ -shelves in the main packing region, but also cut out rectangular slices that serve as shelves for smaller height classes with a full initial buffer. We treat these vertical shelves as usual shelves for their corresponding height class. The width of each vertical main packing shelf for a height class  $H_k$  is always  $2^{-k}$  and its height is always  $1/4$ , as it was formed by a slice of an  $H_2$ -shelf; see Fig. 4 for a sample packing.

Once the packing in a vertical shelf  $\mathcal{V}$  reaches a certain height, we use parts of the buffer region to pack  $H_k$ -squares before continuing to fill  $\mathcal{V}$ ; see details below. When  $\mathcal{V}$  is full, we open a new vertical shelf at the end of the current main packing squares.

*Buffer Usage.* As mentioned above, we alternately use vertical subshelves in the main packing region and space in the buffer region to pack  $H_{\geq 3}$ -squares.

Once the placement of an  $H_k$ -square  $Q$  in the current active vertical  $H_k$ -shelf  $\mathcal{V}$  would intersect with the *head*( $\mathcal{V}$ ) for the first time, we initiate a *buffer packing*. The idea is to make use of the space in the buffer region  $B$  in order to accommodate a large number of very small squares. At the same time we want to avoid collisions with medium squares. Thus, we pack squares in a way that makes the packing in the buffer region and the main packing region grow proportionally. We use the two variables  $\beta$  and  $\alpha$  to guide the growing process.

**Definition 1** Let  $b$  denote the total width of the used sections in  $B$ ,  $e$  denote the number of end buffer regions that have been closed and  $v$  denote the total width of

the vertical shelves in the main packing region. We define  $\beta := b + 1.5/16 \cdot e - 3/16$  and  $\alpha := v/2$ .

The first shelf used for the buffer packing is  $B_1$ . We distinguish the following cases for packing square  $Q$  with side length  $x$ .

1. If  $\beta \geq \alpha$  or  $\beta + x \geq \alpha + 1/16$ , we simply pack square  $Q$  and following  $H_k$ -squares into  $\mathcal{V}$ .
2. Otherwise, we extend the buffer packing as follows:
  - (a) If  $k = 3$ , we pack  $Q$  according to the general shelf-packing concept into the current buffer shelf. We continue by packing next incoming  $H_3$ -squares into  $\mathcal{V}$ .
  - (b) Otherwise ( $k \geq 4$ ), we open a vertical  $H_k$ -shelf  $\mathcal{B}$  at the end of the current buffer packing. We keep on packing incoming  $H_k$ -squares into  $\mathcal{B}$  until the shelf is full and then return to packing  $H_k$ -squares into  $\mathcal{V}$ .

*Buffer Region Overflow.* A packing overflow may also occur in the buffer regions  $B_1, \dots, B_4$ . In this case, we proceed as in every other shelf: we pack the current  $H_3$ -square or buffer subshelf into the next  $B_i$ -region in order. If an  $H_{\geq 4}$ -square cannot be placed at the end of  $usedSection(B_4)$ , we pack it into the end of any of the other buffer regions of  $B \cup E$ . We claim that there must be enough space for this in at least one of the  $B_j$ -regions, as long as the total input area does not exceed  $11/32$ ; see Section 2.5.2.

We also want to make use of the extra space that may remain unoccupied at the end of the main packing regions once the corresponding  $H_2$ -shelf is closed.

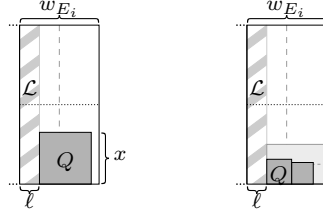
*Additional End Buffer Usage.* The actual buffer space available in the  $E_i$ -regions depends on the lengths of the main packing sections that overlap with them. We call the part of  $usedSection(M_i)$  that overlaps with the considered  $E_i$  region  $\mathcal{L}$ . We denote the width of  $\mathcal{L}$  by  $\ell$ . Depending on  $\ell$ , we choose between two different packing schemes for using  $E_i$  as a buffer.

1. If  $\ell > w_{E_i}/2$ : Close the buffer shelf  $E_i$  (without having actively used it as an  $H_3$ -buffer-shelf).
2. If  $\ell \leq w_{E_i}/2$ : Treat  $E_i \setminus \mathcal{L}$  as a shelf similar to the normal buffer regions  $B_j$ ; see Fig. 5. The only difference is that we pack the shelf vertically and that it may be too narrow to accommodate a large  $H_3$ -square  $Q$ . In that case, we pack  $Q$  into the corresponding active vertical main packing shelf  $\mathcal{V}$  and terminate the buffer packing associated with  $\mathcal{V}$ . We keep packing squares into  $E_i \setminus \mathcal{L}$ , according to the buffer-packing scheme, until the packing reaches a length of  $2/16$  or until we could not fit two of the above mentioned large  $H_3$ -squares into  $E_i$ . We then close  $E_i$ .

We start using (a part of)  $E_i$  as the active buffer region, as soon as the corresponding main shelf  $M_i$  is closed. After  $E_i$  is closed, we continue the buffer packing in the buffer region  $B_i$  that was last active.

### 2.5.2 packSmall Analysis

In this section, we prove that the *packSmall* subroutine successfully packs any sequence of small squares with total area at most  $11/32$ . We argue that an overflow



**Fig. 5** The buffer packing performed in the end buffer regions: (left) packing of a fitting  $H_3$ -square and (right) subshelf packing of  $H_{\geq 4}$  squares.

that is unhandled by the algorithm would only occur if the total area of the input exceeds  $11/32$ . In order to analyze the overall density achieved by the *packSmall* Algorithm, we make some simplifying assumptions on the density reached in the respective shelves. We argue that low density shelves only appear along with high density regions and define a charging scheme that assigns extra areas from dense regions to sparse regions in order to prove the following property.

**Lemma 2** *In any step of the algorithm, the total area of the small squares packed into  $U$  is at least  $\|usedSection(M) \setminus E\|/2$ .*

That is, we use the length of the packing in the main packing region  $M$  as a measure of how much area is occupied by small squares in total. Note that the actual density of the packing in  $M$  may be less. However, we can establish the above result by assigning area occupied in  $A \cup B \cup E$  to  $M$  in the following four different ways.

1. We charge every square  $Q$  that causes an overflow in a shelf  $\mathcal{S}$  with its extra area  $extra(Q)$  and assign it to  $\mathcal{S}$ .
2. We assign a part of the packing in the buffer region to each closed vertical shelf.
3. We associate the occupied area in the initial  $H_k$ -shelf with any open  $H_k$ -shelf.
4. We use the area of the main packing part  $\mathcal{L}_i$  that overlaps with  $E_i$  to form proper buffers in  $E_i$ .

In the following we describe this charging process in more detail.

*Squares Causing an Overflow.* As in the proof of Lemma 1, we charge every  $H_k$ -square  $Q$  that causes an overflow in an  $H_k$ -shelf  $\mathcal{S}$  for leaving a gap at the end of  $\mathcal{S}$ . We assign  $extra(Q)$  to  $\mathcal{S}$  and consider the total  $\tilde{\mathcal{A}}(\mathcal{S})$  of occupied and assigned area of  $\mathcal{S}$  minus the extra area of the squares packed into  $\mathcal{S}$ . By Corollary 1, we get  $\tilde{\mathcal{A}}(\mathcal{S}) \geq \|\mathcal{S} \setminus end(\mathcal{S})\|/2$  for all the shelves that are packed with  $H_k$ -squares only.

Recall that during the packing process we may open vertical shelves in an  $H_k$ -shelf to accommodate a large number of  $H_{>k}$ -squares. In the proof of Lemma 1 we used the fact that  $usedSection(\mathcal{S})$  is always at least half full. This does not generally hold for a shelf  $\mathcal{S}$  that contains vertical shelves. For these shelves, we argue as follows: for every vertical shelf  $\mathcal{V}$  in  $\mathcal{S}$  assign an area of  $\|end(\mathcal{V})\|/2$  from the buffer region to  $\mathcal{V}$ . This way, we get  $\tilde{\mathcal{A}}(\mathcal{V}) \geq \|\mathcal{V}\|/2$  and therefore (by the same argument as in the proof of Lemma 1) we get  $\tilde{\mathcal{A}}(\mathcal{S}) \geq \|\mathcal{S} \setminus end(\mathcal{S})\|/2$  for  $H_k$ -shelves  $\mathcal{S}$  that are packed with  $H_k$ -squares and vertical shelves only which proves the property in Lemma 2.

*Buffer Assignment for Closed Shelves.* As mentioned above, we associate a part of the buffer packing with every vertical shelf  $\mathcal{V}$  in the main packing region. More precisely, we allocate a buffer area of at least  $(w_{\mathcal{V}}/2)^2$  for every vertical shelf  $\mathcal{V}$  with width  $w_{\mathcal{V}}$ . Recall that there are three different cases of initiating a buffer packing with an  $H_k$ -square  $Q$  for a vertical  $H_k$ -shelf  $\mathcal{V}$ ; we either pack  $Q$  into  $\mathcal{V}$  without extending the buffer packing, pack  $Q$  into  $B \cup E$  ( $k = 3$ ) or open a vertical buffer subshelf in  $B \cup E$  ( $k \geq 4$ ).

Depending on which of these options we chose for  $\mathcal{V}$ , we assign it a buffer in one of the following ways:

1. If  $Q$  was packed into  $\mathcal{V}$ , we assign  $extra(Q)$  to the buffer region and a  $w_{\mathcal{V}}/2$ -wide occupied but unassigned slice of the buffer region to  $\mathcal{V}$ .
2. If  $k = 3$  and we packed  $Q$  into the buffer region, we assign an area of  $(w_{\mathcal{V}}/2)^2$  from  $Q$  to  $\mathcal{V}$  and free its extra area (which can be viewed as a buffer slice of width  $x - w_{\mathcal{V}}/2$  and density at least  $1/2$ ) for an assignment to other vertical shelves.
3. If  $k \geq 4$  and we opened a vertical buffer subshelf  $\mathcal{B}$  with  $Q$ , we assign an area of  $(w_{\mathcal{V}}/2)^2$  from  $\mathcal{B}$  to  $\mathcal{V}$  and free the remaining area from the squares in  $\mathcal{B}$  (which is enough to form a  $w_{\mathcal{V}}/2$ -wide buffer slice of density  $1/2$ ) for assignments to other vertical shelves.

With this assignment scheme every vertical main packing shelf  $\mathcal{V}$  with width  $w_{\mathcal{V}}$  effectively allocates a buffer part of total width at most  $w_{\mathcal{V}}/2$ . Note that we only pack  $Q$  into  $\mathcal{S}$  if  $\beta \geq \alpha$  or  $\beta + x \geq \alpha + 1/16$ , where  $\beta$  is a measure for the total length of the buffer packing usable for the assignment to closed vertical shelves and  $\alpha$  is a measure for the total length required. Thus, the buffer slice required for the assignment in case 1. always exists.

**Lemma 3** *Let  $\mathcal{V}$  be a closed vertical shelf for  $H_k$  that has a buffer part assigned to it. Then  $\tilde{\mathcal{A}}(\mathcal{V}) \geq \|\mathcal{V}\|/2$ .*

We make use of the following lemma.

**Lemma 3.1** *Let  $\mathcal{B}$  be a closed vertical buffer subshelf for height class  $H_k$  with width  $w_{\mathcal{B}}$  and height  $h_{\mathcal{B}}$ . Then  $\tilde{\mathcal{A}}(\mathcal{B}) \geq (w_{\mathcal{B}}/2)^2 + w_{\mathcal{B}}/2 \cdot h_{\mathcal{B}}/2$ .*

*Proof.* By construction, we have  $w_{\mathcal{B}} = 2^{-k}$  and height  $h_{\mathcal{B}} = 2^{-3}$ . We open buffer subshelves only for  $H_{\geq 4}$ -squares. Therefore, the height-width ratio of  $\mathcal{B}$  is at least 2, implying  $h_{\mathcal{B}} \geq 2w_{\mathcal{B}}$ . With Corollary 1 we get

$$\begin{aligned} \tilde{\mathcal{A}}(\mathcal{B}) &\geq \frac{\|\mathcal{B} \setminus end(\mathcal{B})\|}{2} \\ &= \frac{(h_{\mathcal{B}} - w_{\mathcal{B}}/2)w_{\mathcal{B}}}{2} \\ &= \left(\frac{h_{\mathcal{B}}}{2} - \frac{w_{\mathcal{B}}}{2}\right) \cdot \frac{w_{\mathcal{B}}}{2} + \frac{h_{\mathcal{B}}}{2} \cdot \frac{w_{\mathcal{B}}}{2} \\ &\geq \left(\frac{w_{\mathcal{B}}}{2}\right)^2 + \frac{h_{\mathcal{B}}}{2} \cdot \frac{w_{\mathcal{B}}}{2} \end{aligned}$$

□

Lemma 3.1 implies that after assigning an area of  $(w_{\mathcal{B}}/2)^2$  to the corresponding vertical main packing shelf of  $H_k$ , we still have at least the area of a  $w_{\mathcal{B}}/2$  by  $h_{\mathcal{B}}$  slice with density  $1/2$  left for the assignment to other vertical shelves.

*Proof. (of Lemma 3)* According to Lemma 3.1 and the buffer assignment scheme, we either assigned a buffer area of  $(w_V/2)^2$  or a  $w_V/2$ -wide buffer slice with density  $1/2$  to  $V$ . By construction, each slice has height  $2^{-3}$ . Let  $B_V$  be the buffer part that we assign to  $V$ . We get  $\tilde{\mathcal{A}}(B_V) \geq \min\{w_V/2 \cdot h_V/2, (w_V/2)^2\} = (w_V/2)^2$ . as  $w_V = 2^{-k} \leq 2^{-3}$  for any  $k \geq 3$ . With Corollary 1 we get in total  $\tilde{\mathcal{A}}(V) \geq \|V\|/2 - (w_V/2)^2 + \tilde{\mathcal{A}}(B_V) \geq \|V\|/2$ .  $\square$

*Buffer Assignment for Open Shelves.* The only vertical shelves for which we have not discussed the buffer assignment so far are open shelves. To account for the gaps remaining in any open vertical  $H_k$ -shelf, we use the area occupied in the initial  $H_k$ -shelf  $\mathcal{I}_k$ .

**Lemma 4** *If  $V$  is an open vertical shelf for  $H_k$ , then  $\tilde{\mathcal{A}}(V) \geq \|V\|/2$ .*

The proof is based on the following properties (Lemmas 4.1-5), which are the results of simple area computations.

**Lemma 4.1** *Let  $\mathcal{I}$  be a closed initial buffer for  $H_k$  with height  $h_{\mathcal{I}}$ . Then,  $\tilde{\mathcal{A}}(\mathcal{I}) \geq (1/4 \cdot h_{\mathcal{I}})/2 - (h_{\mathcal{I}}/2)^2$ .*

*Proof.* Each initial buffer in  $A$  has a width of  $1/4$ . The treatment of the initial  $H_3$ -buffers is also equivalent to filling an  $H_3$ -buffer with width  $1/4$ . Thus, with  $\|\mathcal{I}\| = 1/4 \cdot h_{\mathcal{I}}$  the claim follows directly from Corollary 1.  $\square$

**Lemma 4.2** *Let  $V$  be an open vertical main packing shelf and  $\mathcal{I}$  the initial buffer for height class  $H_k$  with  $k \geq 3$ . Then:  $\tilde{\mathcal{A}}(\mathcal{I}) + \tilde{\mathcal{A}}(V) \geq \|V\|/2$ .*

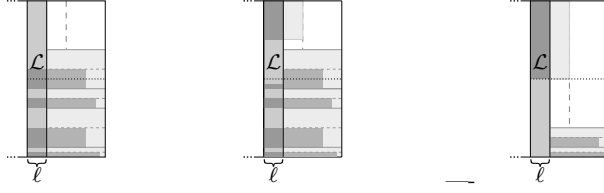
*Proof.* We know that, by construction, the width  $w_V$  of  $V$  equals  $2^{-k}$  and the height  $h_V$  equals  $1/4$ . Each vertical shelf is opened with a square of the corresponding height class  $H_k$  packed into it. Because each square of  $H_k$  has a side length  $> 2^{-(k+1)} = w_V/2$ , we get  $\tilde{\mathcal{A}}(V) > (w_V/2)^2$ . By construction of the shelves, the height  $h_{\mathcal{I}}$  of the initial buffer  $\mathcal{I}$  equals  $2^{-k} = w_V$ . Thus, with Lemma 4.1 we get  $\tilde{\mathcal{A}}(\mathcal{I}) + \tilde{\mathcal{A}}(V) \geq (1/4 \cdot h_{\mathcal{I}})/2 - (h_{\mathcal{I}}/2)^2 + (w_V/2)^2 = (h_V \cdot w_V)/2 = \|V\|/2$ .  $\square$

We may have an open vertical buffer subshelf and an open vertical main shelf at the same time. In that case we use the area in the initial  $H_k$ -shelf to close the gaps in both vertical shelves.

**Lemma 4.3** *Let  $V$  be an open vertical main packing shelf and  $\mathcal{I}$  be the initial buffer shelf for height class  $H_k$  with  $k \geq 4$ . If there is an open buffer subshelf  $\mathcal{B}$  for  $H_k$ , then  $\tilde{\mathcal{A}}(\mathcal{I}) + \tilde{\mathcal{A}}(V) + \tilde{\mathcal{A}}(\mathcal{B}) \geq \|V\|/2 + \|\mathcal{B}\|/2$ .*

*Proof.* We only open a buffer subshelf  $\mathcal{B}$  with a square  $Q$  that would have intersected  $\text{head}(V)$  if it was packed into the main packing shelf  $V$ . Hence, it must hold:  $\tilde{\mathcal{A}}(V) + \tilde{\mathcal{A}}(\mathcal{B}) \geq (\|V\| - w_V^2)/2$ . Note, that we have  $h_V = w_{\mathcal{I}} = 2h_{\mathcal{B}} = 2^{-2}$  and  $w_V = h_{\mathcal{I}} = w_{\mathcal{B}} \leq 2^{-4}$  which implies  $w_V \leq h_V/4$ . Additionally, we know  $\tilde{\mathcal{A}}(\mathcal{I}) \geq 3/8 \cdot \|\mathcal{I}\|$  by Corollary 1. Thus, in total we get:

$$\begin{aligned} \tilde{\mathcal{A}}(\mathcal{I}) + \tilde{\mathcal{A}}(V) + \tilde{\mathcal{A}}(\mathcal{B}) &\geq \frac{\|V\| - w_V^2}{2} + \frac{3}{8} \cdot w_{\mathcal{I}} h_{\mathcal{I}} \\ &= \frac{\|V\|}{2} + \frac{3w_V h_V - 4w_V^2}{8} \\ &\geq \frac{\|V\|}{2} + \frac{w_V h_V}{4} = \frac{\|V\|}{2} + \frac{\|\mathcal{B}\|}{2} \end{aligned}$$



**Fig. 6** The assignment of  $\tilde{\mathcal{A}}(\mathcal{L})$  when  $E_i$  is closed: (left) we use all of  $\tilde{\mathcal{A}}(\mathcal{L})$  to extend the horizontal subshelves to a total length of  $w_{E_i}$ , (center) we use one half of  $\tilde{\mathcal{A}}(\mathcal{L})$  for the horizontal subshelf extensions and the other half for an  $H_3$ -buffer or (right) we use all of  $\tilde{\mathcal{A}}(\mathcal{L})$  for (two)  $H_3$ -buffer assignments.

□

By construction we have:

**Lemma 4.4** *There is at most one open vertical main packing shelf and at most one open buffer subshelf for any height class  $H_k$  with  $k \geq 3$  in each step of the algorithm.*

Consequently, we have to account for the space in only one vertical main packing shelf and at most one buffer subshelf at a time.

*Proof. (of Lemma 4)* We allocate an initial buffer for each height class  $H_k$ , before opening vertical  $H_k$ -shelves in  $M$ . Thus, by Lemma 4.4 there is an initial buffer assigned to each open vertical shelf and the claim follows directly with Lemmas 4.2 and 4.3. □

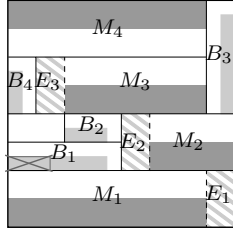
*End Buffer Assignment.* Recall that the packing in the main packing region  $M_i$  may reach into the respective ends  $E_i$  and thus effect the usage of  $E_i$  as a buffer region. Let  $\mathcal{L}_i$  be the part of the main packing that overlaps with  $E_i$ . We argue that the area  $\tilde{\mathcal{A}}(\mathcal{L}_i)$  of  $\mathcal{L}_i$  is sufficient to extend the buffer packing in  $\mathcal{L}_i \setminus E_i$  to a full width of  $2/16$  (the same as the height of  $B$ ). This way we can use  $E_i$  analogously to the buffer regions  $B_j$  when assigning buffers to vertical main packing shelves. Because we use a buffer of total length at most  $0.5/16$  for  $\mathcal{L}_i$ , we gain additional buffers of total length at least  $1.5/16$  from  $E_i$  for the assignment to  $M \setminus E$ .

**Lemma 5** *We get an additional buffer length of  $1.5/16$  for  $M \setminus E$  from  $E_i$ .*

*Proof.* Let  $\mathcal{L}_i$  be the part of  $\text{usedSection}(M_i)$  overlapping with  $E_i$  and  $\ell_i$  be the width of  $\mathcal{L}_i$ . By assumption we have,  $\tilde{\mathcal{A}}(\mathcal{L}_i) \geq \|\mathcal{L}_i\|/2 = (\ell_i \cdot 1/4)/2$ . In the end buffer region  $E_i$ , we consider the buffer length gained in  $E_i$  minus the buffer used for  $\mathcal{L}_i$  as the additional buffer for  $M \setminus E$ . We distinguish two different cases for the width  $\ell_i$  of the overlap:

1.  $\ell_i > w_{E_i}/2$ : We simply close  $E_i$  without any actual buffer packing. The buffer area used for  $\mathcal{L}_i$  is at most  $\ell_i/2 \cdot 1/16$ . Therefore, the extra buffer area constructed in  $E_i$  is at least:

$$\begin{aligned} \tilde{\mathcal{A}}(\mathcal{L}_i) - \frac{\ell_i/2}{16} &\geq \frac{2\ell_i}{16} - \frac{\ell_i/2}{16} \\ &= \frac{3/2 \cdot \ell_i}{16} > \frac{1.5}{16^2} \end{aligned}$$



**Fig. 7** Total Buffer Length Consideration. Dark gray: main packing section that requires a buffer assignment. Light gray: usable buffer space, including the area for the initial  $H_3$ -buffer (marked by a cross)

This is sufficient to form buffer slices with a total length of  $1.5/16$  and density  $1/2$ .

2.  $\ell_i \leq w_{E_i}/2$ : We consider the buffer assignment to a vertical shelf  $\mathcal{V}$  that initiated a buffer packing with a square  $Q$  when the current active buffer was the end buffer  $E_i$ . If  $Q$  was a large  $H_3$ -square that did not fit into  $E_i \setminus \mathcal{L}_i$ , then  $\text{extra}(Q)$  and  $\tilde{\mathcal{A}}(\mathcal{L}_i)/2$  to form a buffer of area  $(1/16)^2 = (w_{\mathcal{V}}/2)^2$  for  $\mathcal{V}$ . The case when  $Q$  was packed into  $E_i$  is handled analogously to the assignment of buffers from  $B$ ; take an area of  $(w_{\mathcal{V}}/2)^2$  from  $Q$  and free the remainder for assignment to other vertical shelves. We use any area left over from  $\tilde{\mathcal{A}}(\mathcal{L}_i)$  to extend potential buffer subshelves to full width  $1/8$ ; see Fig. 5. In total, we form proper buffers with of a combined length of at least  $2/16$ . Because  $\mathcal{L}_i$  itself uses a buffer length of  $\ell_i/2$ , we get additional buffers with a total length of at least  $2/16 - \ell_i/2 \geq 2/16 - 0.5/16 = 1.5/16$  for the assignment to vertical shelves in  $M \setminus E$ .

□

With the extra buffer length gained in  $E$ , we provide enough buffer space to fill the gaps remaining in all vertical main packing shelves in  $M \setminus E$ .

*Sufficiency of the Buffer Area.* We prove that the buffer regions  $A \cup B \cup E$  are large enough to provide buffer space for all vertical shelves in  $M \setminus E$ .

**Lemma 6** *The total space provided in  $A \cup B \cup E$  is sufficient for the buffer packing of vertical shelves in  $M \setminus E$ .*

We first establish a series of bookkeeping lemmas.

**Lemma 6.1** *The total buffer length required for  $M \setminus E$  is at most  $22/16$ .*

*Proof.* The main packing region  $M_1, \dots, M_4$  have a total length of  $1 + 1/2 + 3/4 + 7/8 = 25/8$ ; see Fig. 7. We do not need buffer space for the end regions  $E_1$ ,  $E_2$  and  $E_3$ , which have a total length of  $3 \cdot 1/8$ . Thus, we require buffer sections with a total length of at most  $1/2 \cdot (25/8 - 3/8) = 11/8$ . □

**Lemma 6.2** *The total length of buffer area in  $B$  usable for the assignment to closed vertical main packing shelves is at least  $17/16$ .*

*Proof.* The buffer areas  $B_1, \dots, B_4$  have a total length of  $1/2 + 1/4 + 1/2 + 1/4 = 6/4$ ; see Fig. 7. We can only guarantee  $B_i \setminus \text{end}(B_i)$  to be successfully used as buffer area (again see Corollary 1). In addition, count  $3/16$  towards the initial buffer for  $H_3$ . In sum, we provide a total buffer length of  $17/16$  in  $B$ . □

Recall that we start using the ends  $E_i$  as a buffer region only when the  $H_2$ -shelf  $M_i$  is closed. Thus, we have to make sure that the remaining buffer space is sufficient before  $E_i$  is available.

**Lemma 6.3** *We do not need the additional buffer area of  $E_i$  before the main packing in  $M_i$  overflows.*

*Proof.* We know that  $B$  provides enough buffer area for  $M_1 \cup M_2$ ; see Lemma 6.2. Until  $E_3$  is used as a buffer region, we provide the required buffer with  $B \cup E_1 \cup E_2$ . Before  $M_3$  overflows, we require buffer area for at most  $M_1 \setminus E_1 \cup M_2 \setminus E_2 \cup M_3 \cup \text{usedSection}(M_4)$ . As soon as  $\text{length}(\text{usedSection}(M_4)) \geq 3/8$ , we keep using  $M_3$  as the main packing region. At that time, the last part of  $\text{usedSection}(M_4)$  is either occupied by a vertical main shelf, then  $\text{length}(\text{usedSection}(M_4)) \leq 4/8$ , as no vertical shelf is wider than  $1/8$  or it is occupied by an  $H_2$ -square, then we do not need to assign a buffer to this part of  $\text{usedSection}(M_4)$ . That is, in total we require a buffer length of  $1/2 \cdot (7/8 + 3/8 + 6/8 + 4/8) = 20/16$ . By Lemmas 6.2 and 5 we get a buffer length of at least  $17/16 + 2 \cdot 1.5/16 = 20/16$  from  $B \cup E_1 \cup E_2$ .  $\square$

**Lemma 6.4** *If a square of  $H_{\geq 5}$  does not fit into any of the buffer regions in  $B \cup E$ , then the total buffer length gained from  $B \cup E$  is larger than  $22/16$ .*

*Proof.* By definition, any square  $Q$  of  $H_{\geq 5}$  has a side length  $x \leq 2^{-5} = 1/32$ . Consequently, the total length of the used sections in  $B_1, \dots, B_4$  must be larger than the total length of  $B$  minus  $4 \cdot 2^{-5}$ . As each occupied slice of  $B$  can be used for the buffer assignment to any height class, we provided a total usable buffer length of  $1/2 + 1/4 + 1/2 + 1/4 - 4 \cdot 1/32 = 11/8$ . A part with length at most  $3/16$  is used for the initial buffer of  $H_3$ . From  $E$  we get a buffer length of  $4.5/16$ ; see Lemma 5. Thus, we provide a total buffer length of more than  $11/8 - 3/16 + 4.5/16 = 23.5/16$  for the buffer assignment to closed shelves.  $\square$

With the help of Lemmas 6.1 to 6.3 we are able to prove that the provided buffer area suffices.

*Proof. (of Lemma 6)* By Lemma 6.1 it suffices to occupy buffer with a total length of  $22/16$  for the vertical shelves in  $M$ . The initial buffer parts account for the gaps remaining in open shelves; see Lemma 4.2. By Lemmas 6.2 and 5, we gain an additional buffer length of  $21.5/16$  for closed shelves from  $B \cup E$ . Let  $w$  be the total width of all the shelves that remain open at the time the subroutine terminates. Then the total provided buffer length is at least  $w/2 + 21.5/16$ . This is at least  $22/16$  if at least one vertical  $H_3$ - or  $H_4$ -shelf exists in  $M$  (as we only close a vertical main shelf when simultaneously opening a new one). Otherwise, we created vertical shelves only for  $H_{\geq 5}$ -squares in  $M$ . In Lemma 6.4 we proved that we can fit  $H_{\geq 5}$ -squares into the buffer region until sufficient buffer space ( $23.5/16$ ) is provided. Thus, in both cases the provided buffer space is sufficient.  $\square$

*Overall Density.* Lemmas 3, 4 and 6 directly prove the invariant of Lemma 2. That is, we can w.l.o.g. assume a density of  $1/2$  for the used sections of  $M \setminus E$ . By construction, the algorithm successfully packs all small squares, until a square  $Q$  would intersect the top left corner of  $U$  in  $M_4$ . At this time the total area of small input squares must be greater than  $\|\bigcup_{i=1}^3 (M_i \setminus E_i) \cup M_4\|/2 = 11/32$ .

**Theorem 2** *The packSmall Subroutine packs any sequence of small squares with total area at most  $11/32$  into the unit square.*



*Some Additional Properties.* Before we analyze the algorithms performance in the presence of large and medium squares, we state a couple of important properties of the packing created with small squares.

Recall that we use variable  $\beta$  (see Def. 1) to restrict the growth of the buffer packing. By construction, we can relate the length of the buffer region and the total area of the input as follows.

**Lemma 7** *Let  $Q$  be a small square with side length  $x$  in the buffer region  $B$  and let  $\mathcal{P}_s$  be the set of small squares received so far. Then the total area of the small input squares  $\|\mathcal{P}_s\|$  is greater than  $(\beta + x - 1/16) \cdot 1/4$ .*

*Proof.* Let  $H_k$  be the height class of  $Q$ . Because we never pack  $H_2$ -squares into the buffer region, we have  $k \geq 3$ . By construction, we only pack an  $H_3$ -square into  $B$  if  $\beta + x < \alpha + 1/16$ . We open a buffer subshelf for  $Q$  only if  $k \geq 4$  and  $\beta < \alpha$ . In this case we have  $x \leq 1/16$ . Thus, independent of  $k$ , we get  $\beta + x - 1/16 < \alpha$ . Recall that  $\alpha$  is defined as the total width of the vertical shelves in  $M$ . Let  $\Sigma$  be the union of all the vertical shelves in  $M$ . Thus, with Lemmas 3 and 4 we get

$$\begin{aligned} \|\mathcal{P}_s\| &\geq \tilde{\mathcal{A}}(\Sigma) \geq \|\Sigma\|/2 \\ &= (\alpha \cdot 1/4) \\ &> (\beta + x - 1/16) \cdot 1/4 \end{aligned}$$

□

As a direct implication of Lemma 7 and the fact that when  $B_4$  is first used as the active buffer region, both additional buffer regions  $E_1$  and  $E_2$  have successfully been used and closed by the algorithm before, we get the following lower bounds for the total area  $\|\mathcal{P}_s\|$  of small squares packed, as a function of the total length of the packing in  $B$ .

*Property 1* Let  $Q$  be a small square in the buffer region  $B_2$  with side length  $x$  and distance  $d > 1/4$  to the left boundary of  $U$ , then  $\|\mathcal{P}_s\| > (d + x - 1/16) \cdot 1/4$ .

*Property 2* If there is a small square in  $B_3$ , Then  $\|\mathcal{P}_s\| > 7/64$ .

*Property 3* Let  $Q$  be a small square in  $B_3$  with side length  $x$  that was packed in a distance  $d > 0$  to the bottom of  $B_3$ . Then  $\|\mathcal{P}_s\| > (7.5/16 + d + x) \cdot 1/4$ .

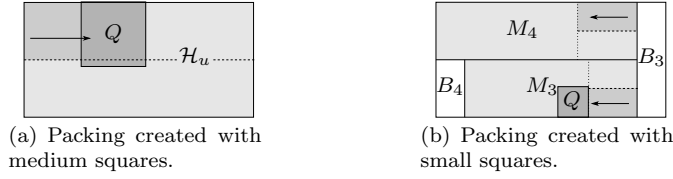
*Property 4* If there is a small square in  $B_4$ , then  $\|\mathcal{P}_s\| > 17/64$ .

*Property 5* Let  $Q$  be a small square in  $B_4$  with side length  $x$  and distance  $d > 0$  from the bottom of  $B_4$ . Then  $\|\mathcal{P}_s\| > (1 + d + x) \cdot 1/4$ .

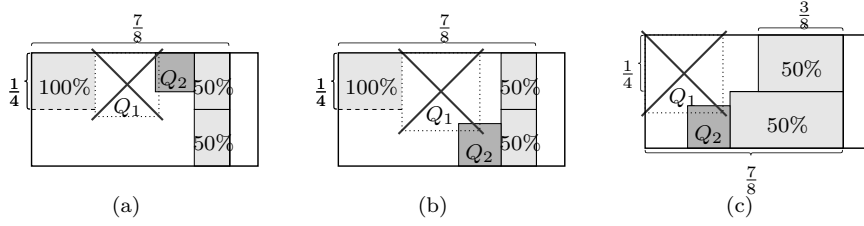
The following properties follow directly from Lemma 2.

*Property 6* When the first small square is packed into  $M_2$ , then  $\|\mathcal{P}_s\| \geq 7/64$ .

*Property 7* When the first small square is packed into  $M_3$ , then  $\tilde{\mathcal{A}}(\mathcal{H}_\ell) \geq 10/64$ .



**Fig. 8** Packing performed in the upper half of  $U$ . The feasible packing area has light gray background, the medium gray part represents the packing created before  $Q$  was placed.



**Fig. 9** Collision of a medium square  $Q_1$  with an  $H_2$ -square  $Q_2$  in  $\mathcal{H}_u$ .

## 2.6 Combined Analysis

In the previous sections we proved that the algorithm successfully packs large, medium and large squares separately, as long as input has a total area of at most  $11/32$ . A case distinction over all possible collisions that may appear between the packings of these height classes can be used to prove the main result.

**Theorem 3** *The Recursive Shelf Algorithm packs any sequence of squares with total area at most  $11/32$  into the unit square.*

We prove the claim by showing that if the algorithm fails to pack a square, the total area of the given squares must exceed  $11/32$ . In the following we analyze the packing density at the time a collision of the different packing subroutines would appear. First we consider a collision between a medium and a small square in the upper half  $\mathcal{H}_u$  of the unit square container.

**Lemma 8** *If a medium square  $Q_1$  collides with a part of the packing constructed with small squares, then  $\|Q_1\| + \tilde{\mathcal{A}}(\mathcal{H}_u) \geq 6/32$ .*

*Proof.* Recall that we pack the medium sized squares from left to right aligned with the top boundary of  $\mathcal{H}_u$ ; see Fig. 8(a). The packing of small squares (into  $M_3$  and  $M_4$ ) is performed from right to left; see Fig. 8(b). Also recall that we alternately use  $M_3$  and  $M_4$  as the current main packing region (choosing which ever half is less full in width) until the packing in  $M_4$  reaches a total length at least  $3/8$ . Then we only pack  $M_3$  until it is completely filled, before finishing the packing in  $M_4$ .

Let  $Q_1$  be a medium square that collides with a small square in the upper half  $\mathcal{H}_u$  of the unit bin. Then  $Q_1$  either intersects a vertical shelf  $\mathcal{S}$  or an  $H_2$ -square  $Q_2$ . The main idea is to prove that the parts of  $\mathcal{H}_u \setminus B_3$  both right and left to

$Q_2/\mathcal{S}$  have a density of  $1/2$ . We distinguish six different cases depending on the location of  $\mathcal{S}$  or  $Q_2$  in  $\mathcal{H}_u$ .

1.  $Q_1$  collides with an  $H_2$ -square  $Q_2$  in  $M_4$ :

We know  $\text{length}(\text{usedSection}(M_3)) > \text{length}(\text{usedSection}(M_4)) - w_S$ , as otherwise we would have packed  $Q_2$  in  $M_3$ . Therefore, the entire part of  $M_3 \cup M_4$  right to  $Q_2$  must be used by small squares, thus having a density of  $1/2$ . Additionally, the section used by the  $H_1$ -squares must be filled to a height of at least  $1/4$ . Hence, we know that the sections of  $\mathcal{H}_u \setminus B_3$  both right and left to  $Q_2$  are half full; see Fig. 9(a). Therefore, with  $x_2 \geq 1/8$ ,

$$\begin{aligned} \tilde{\mathcal{A}}(\mathcal{H}_u) &> \frac{(7/8 - x_2) \cdot 1/2}{2} + x_2^2 \\ &\geq \frac{7}{32} + x_2 \left( x_2 - \frac{1}{4} \right) \\ &\geq \frac{7}{32} + \frac{1}{8} \cdot \left( \frac{1}{8} - \frac{1}{4} \right) > \frac{6}{32}. \end{aligned}$$

2.  $Q_1$  collides with an  $H_2$ -square  $Q_2$  in  $M_3$ :

By construction we have  $\text{length}(\text{usedSection}(M_3)) - x_2 \leq \text{length}(\text{usedSection}(M_4))$  or  $\text{length}(\text{usedSection}(M_4)) \geq 3/8$ , as we packed  $Q_2$  into  $M_3$  instead of  $M_4$ .

- (a) If  $\text{length}(\text{usedSection}(M_3)) - x_2 \leq \text{length}(\text{usedSection}(M_4))$ , the situation is symmetric to the one in the previous case; see Fig. 9(b). Because  $Q_1$  is aligned with the top and  $Q_2$  with the bottom of  $\mathcal{H}_u$  and  $Q_1$  and  $Q_2$  collide, we have  $x_1 + x_2 > 1/2$ . Thus, we get

$$\begin{aligned} \tilde{\mathcal{A}}(\mathcal{H}_u) &> \frac{(7/8 - x_1 - x_2) \cdot 1/2}{2} + x_1^2 + x_2^2 \\ &\geq \frac{7}{32} - \frac{x_1 + x_2}{4} + \frac{(x_1 + x_2)^2}{2} \\ &> \frac{7}{32} - \frac{(x_1 + x_2) \cdot 1/2}{2} + \frac{(x_1 + x_2) \cdot 1/2}{2} > \frac{6}{32}. \end{aligned}$$

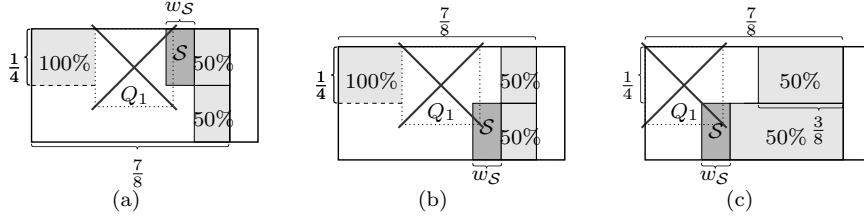
- (b) Otherwise,  $\text{length}(\text{usedSection}(M_3)) - x_2 > \text{length}(\text{usedSection}(M_4)) \geq 3/8$ ; see Fig. 9(c). Again, we know  $x_1 + x_2 > 1/2$ . Thus,

$$\begin{aligned} \tilde{\mathcal{A}}(\mathcal{H}_u) &\geq x_1^2 + x_2^2 + \frac{\|\text{usedSection}(M_3)\|}{2} + \frac{\|\text{usedSection}(M_4)\|}{2} \\ &> \frac{(x_1 + x_2)^2}{2} + 2 \cdot \frac{\|\text{usedSection}(M_4)\|}{2} \\ &> \frac{(1/2)^2}{2} + \frac{3}{8} \cdot \frac{1}{4} > \frac{6}{32}. \end{aligned}$$

3.  $Q_1$  collides with a vertical shelf  $\mathcal{S}$  in  $M_4$ :

Analogously to the first case, we know that the sections of  $\mathcal{H}_u \setminus B_4$  both right and left to  $\mathcal{S}$  must be half full; see Fig. 10(a). Thus, with  $w_S \leq 1/8$  we get:

$$\begin{aligned} \tilde{\mathcal{A}}(\mathcal{H}_u) &> \frac{(7/8 - w_S) \cdot 1/2}{2} + \frac{\|\mathcal{S}\|}{2} \\ &\geq \frac{7}{32} - \frac{w_S}{4} \geq \frac{6}{32}. \end{aligned}$$



**Fig. 10** Collision of a medium square  $Q_1$  with vertical shelf  $S$  in  $H_u$ .

4.  $Q_1$  collides with a vertical shelf  $S$  in  $M_3$ :

Analogously to the second case, we must have  $\text{length}(\text{usedSection}(M_3)) - w_S \leq \text{length}(\text{usedSection}(M_4))$  or  $\text{length}(\text{usedSection}(M_4)) \geq 3/8$  as we opened  $S$  in  $M_3$ .

- (a) If  $\text{length}(\text{usedSection}(M_3)) - w_S \leq \text{length}(\text{usedSection}(M_4))$ , then we have the same conditions as described in the first case; see Fig. 10(b). We analogously get

$$\tilde{\mathcal{A}}(\mathcal{H}_u) > \left(\frac{7}{8} - w_S\right) \frac{1}{4} \geq \frac{6}{32}.$$

- (b) Otherwise,  $\text{length}(\text{usedSection}(M_3)) - w_S > \text{length}(\text{usedSection}(M_4)) \geq 3/8$ . Let  $\ell$  be the length of the  $H_u$ -section used by  $H_1$ . We know  $\ell \geq 1/4$  and  $\text{length}(\text{usedSection}(M_3)) > 7/8 - \ell$ ; see Fig. 10(c). Thus,

$$\begin{aligned} \tilde{\mathcal{A}}(\mathcal{H}_u) &\geq \ell \cdot \frac{1}{4} + \frac{\|\text{usedSection}(M_3)\|}{2} + \frac{\|\text{usedSection}(M_4)\|}{2} \\ &> \ell \cdot \frac{1}{4} + \frac{(7/8 - \ell) \cdot 1/4}{2} + \frac{3/8 \cdot 1/4}{2} \\ &\geq \frac{\ell}{8} + \frac{7}{64} + \frac{3}{64} \geq \frac{6}{32}. \end{aligned}$$

□

We are now able to prove Theorem 3.

*Proof. (of Theorem 3)* Let  $Q$  be the square at which the algorithm stops. Denote  $\sigma$  the set of all input squares and  $\mathcal{P}$  the set of all squares packed at the time  $Q$  arrives. We claim  $\|Q\| + \|\mathcal{P}\| > 11/32$ . To prove this statement we distinguish the different types of collisions that might cause the algorithm to stop with failure.

1. *The input sequence  $\sigma$  consists of squares belonging to  $H_0$  only:*

In this case, the algorithm successfully packs the first square into the top right corner and then fails to pack the second square  $x$ . Because both squares belong to  $H_0$ , we have  $\|\sigma\| = \|Q\| + \|\mathcal{P}\| > 2 \cdot (1/2)^2 = 1/2 > 11/32$ .

2. *The input sequence  $\sigma$  consists of squares belonging to  $H_1$  only:*

The algorithm successfully packs all the squares in  $\sigma$  according to the Ceiling Packing Subroutine, unless the total area of  $\sigma$  is greater than  $3/8 > 11/32$ ; see Theorem 1.

3. *The input sequence  $\sigma$  consists of small squares only:*

The algorithm successfully packs all the squares in  $\sigma$  according to the packS-small subroutine, unless the total area of  $\sigma$  is greater than  $11/32$ ; see Theorem 2.

4. *A large square  $Q_0$  collides with a medium square  $Q_1$ :*

In this case, the first (and only) square of  $H_0$  collides with the L-shaped packing produced by the Ceiling Algorithm; see Fig. 11(b). We know  $\|Q_0\| > (1/2)^2 = 1/4$  and the shelf packing for the  $H_1$ -squares must reach from the left boundary to more than a distance of  $x_0$  from the right boundary. Thus, as  $x_i \geq 1/4$  for any square  $Q_i \in H_1$ , the total area of the input sequence  $\|\sigma\|$  is at least  $\|\mathcal{P}\| + \|Q_0\| > x_0^2 + (1 - x_0) \cdot \frac{1}{4} \geq 3/8 > 11/32$ .

5. *A large square  $Q_0$  collides with a small square  $Q_s$ :*

If the side length  $x_0$  of  $Q_0$  is greater than  $\sqrt{11/32}$ , then  $\|Q_0\| > 11/32$  and we do not even need a small square. Therefore, we assume  $x_0 \leq \sqrt{11/32} < 5/8$ . There are two cases:

(a)  *$Q_s$  is in the main packing area:*

Because  $x_0 < 5/8 < 3/4$ ,  $Q_s$  must have been packed into  $M_2$ ,  $M_3$  or  $M_4$ . In any case, a small square must be in  $M_2$  and by Property 6 we have a total area of more than  $7/64$  from small squares. Additionally, we have  $x_0 \geq 1/2$ , as  $Q_0$  is large. That is,  $\|\sigma\| \geq \|Q_0\| + \|\mathcal{P}\| > (1/2)^2 + 7/64 = 23/64 > 11/32$ ; Fig. 11(c).

(b)  *$Q_s$  is in the buffer area:*

W. l. o. g.,  $Q_s$  is located in  $B_2$  (as otherwise  $Q_s$  in  $B_3$  or  $B_4$  and  $B_2$  hence full). Let  $d$  be the distance of  $Q_s$  to the left boundary of the unit square. As  $Q_0$  and  $Q_s$  collide, we have  $d + x_s + x_0 > 1$ ; see Fig. 11(d). We distinguish two cases for the side length of  $Q_0$ :

- i.  $x_0 \in (1/2, 9/16)$ : Then  $d + x_s > 1 - x_0 > 7/16$ , which implies  $d > 7/16 - 1/8 > 1/4$ . Thus, by Property 1 we get

$$\|\mathcal{P}\| > \frac{d + x_s - 1/16}{4} > \frac{7/16 - 1/16}{4} = 6/64$$

- ii.  $x_0 \in [9/16, 5/8)$ : Then  $d + x_s > 1 - x_0 > 3/8$ , which implies  $d > 3/8 - 1/8 = 1/4$ . Thus, by Property 1 we get

$$\|\mathcal{P}\| > \frac{d + x_s - 1/16}{4} > \frac{3/8 - 1/16}{4} = 5/64$$

In total we get

$$\|\sigma\| \geq \|Q_0\| + \|\mathcal{P}\| > \min\{(1/2)^2 + 6/64, (9/16)^2 + 5/64\} = 11/32.$$

6. *A medium square  $Q_1$  collides with a small square  $Q_s$ :*

There are many different types of collisions that might appear between the small square packing and a squares of  $H_1$ . Note that the ceiling packing never interacts with the buffer area of  $\mathcal{H}_\ell$ , but might interact with the buffers in  $\mathcal{H}_u$ .

(a)  *$Q_s$  is (a buffer square) in  $B_3$ :*

Recall that all medium squares are packed from left to right aligned with the top boundary of  $U$ . Let  $d$  be the distance of  $Q_s$  to the lower boundary of  $B_3$ .

We distinguish two cases for  $x_s + d$ :

- i.  $x_s + d \leq 1/8$ : Then  $Q_1$  intersects the  $1/8$ -high section at the bottom of  $B_3$ ; see Fig. 11(e). That is, either an overflow of  $H_1$ -squares occurred in  $\mathcal{H}_u$  and we have a total area of at least  $1 \cdot 1/4$  covered with medium squares, or  $Q_1$  coincides with the top of  $U$ , hence  $x_1 > 3/8$  and the total area of the medium squares is greater than

$$x_1^2 + \left(\frac{7}{8} - x_1\right) \cdot \frac{1}{4} \geq \frac{7}{32} + x_1 \cdot \left(x_1 - \frac{1}{4}\right) \geq \frac{14}{64} + \frac{3}{8} \cdot \frac{1}{8} = \frac{17}{64} > \frac{1}{4}$$

. As  $Q_s$  is in  $B_3$  we get  $\|\mathcal{P}\| > 7/64$  by Property 2. In both cases, we get

$$\|\sigma\| \geq \|Q_1\| + \|\mathcal{P}\| > 1/4 + 7/64 = 11/32.$$

ii.  $x_s + d > 1/8$ : This case is depicted in Fig. 11(f). By Property 3 we get

$$\|\mathcal{P}\| > \frac{7.5/16 + d + x_s}{4} > \frac{7.5/16 + 2/16}{4} = \frac{9.5}{64}.$$

Because  $Q_1$  intersects with  $B_3$ , we know that the total length of the medium square packing is greater than  $7/8$ . Thus we get

$$\|\sigma\| \geq \|Q_1\| + \|\mathcal{P}\| > 7/8 \cdot 1/4 + 9.5/64 > 23.5/64 > 11/32.$$

(b)  $Q_s$  is (a buffer square) in  $B_4$ :

Recall that we start packing medium squares coinciding with the top of  $U$ . We fill the buffer region  $B_4$  from bottom to top. Let  $d$  be the distance of  $Q_s$  to the lower boundary of  $B_4$ . We distinguish two cases for  $x_s + d$ :

i.  $x_s + d \leq 1/8$ : Then  $Q_1$  perturbs the  $1/8$ -high section at the bottom of  $B_4$ , i.e. we have  $x_1 > 3/8$ . Because  $Q_s$  is in  $B_4$ , we get  $\|\mathcal{P}\| > 17/64$  by Property 4. Thus, in total we have

$$\|\sigma\| \geq \|Q_1\| + \|\mathcal{P}\| > (3/8)^2 + 17/64 = 26/64 > 11/32.$$

ii.  $x_s + d > 1/8$ : By Property 5 we have

$$\|\mathcal{P}\| > \frac{1 + d + x_s}{4} > \frac{1 + 1/8}{4} = \frac{9}{32}.$$

Because  $Q_1$  is a medium square, we have  $x \geq 1/4$  and get

$$\|\sigma\| \geq \|Q_1\| + \|\mathcal{P}\| > (1/4)^2 + 9/32 = 11/32.$$

(c)  $Q_s$  is packed into  $M_3$  or  $M_4$ :

By Property 7 and Lemma 8 we have  $\tilde{\mathcal{A}}(\mathcal{H}_\ell) \geq 5/32$  and  $\tilde{\mathcal{A}}(\mathcal{H}_u) \geq \tilde{\mathcal{A}}(M_3 \setminus E_i \cup M_4) \geq 6/32$ , respectively. Therefore,  $\tilde{\mathcal{A}}(U) \geq 11/32$ .

(d)  $Q_s$  is a buffer square in  $E_i$ :

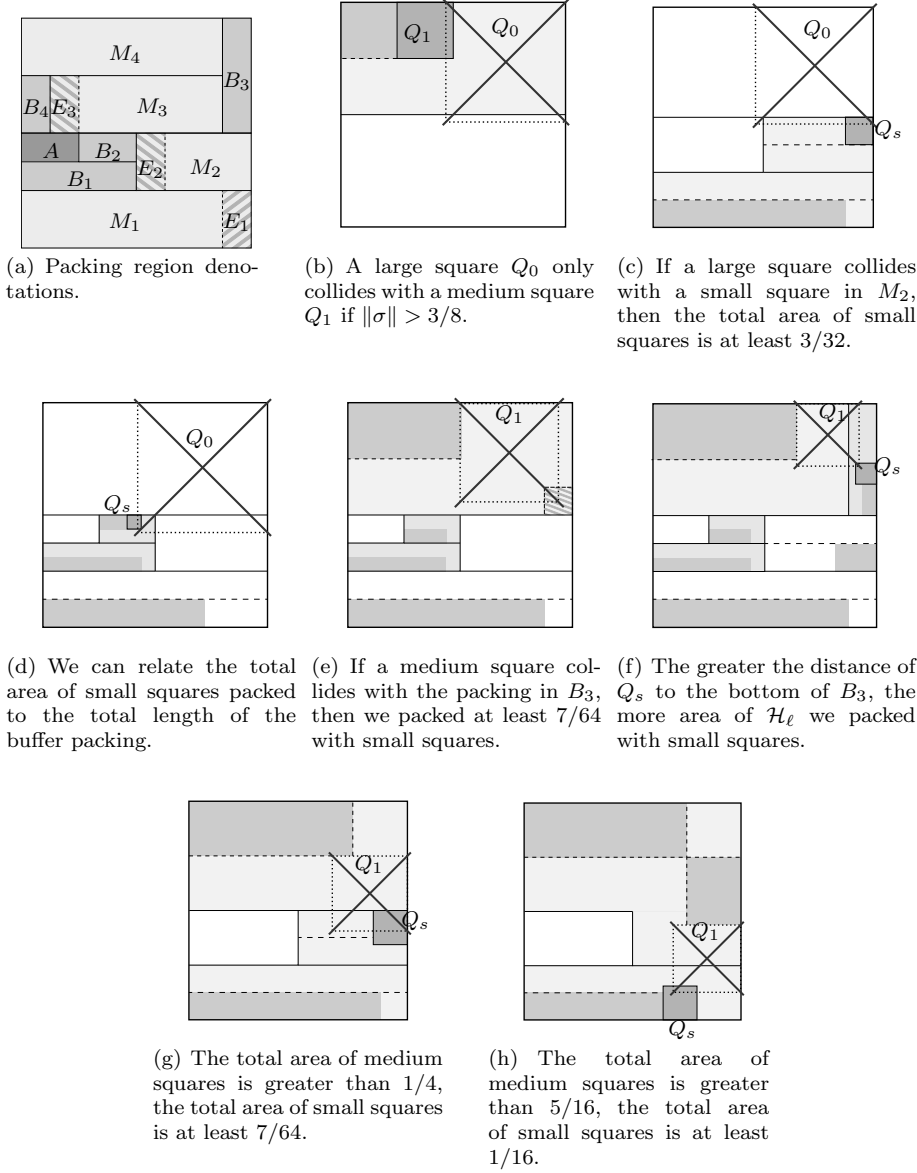
We start treating the end  $E_i$  of a main packing area  $M_i$  only if  $M_i \setminus E_i$  is fully used. Therefore, this type of collision can be handled analogously to the collision of  $Q_0$  with a square in  $M_i$ .

(e)  $Q_1$  overlaps with  $M_2$  but not with  $M_1$ :

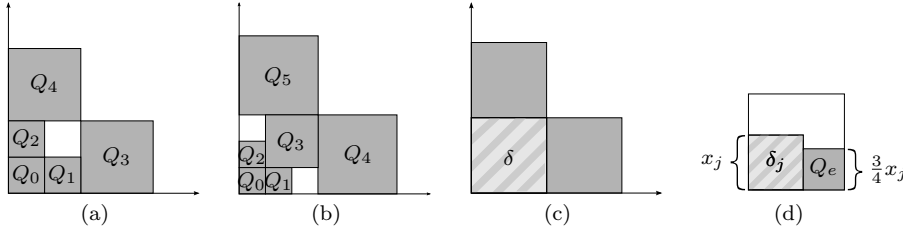
This only happens if  $Q_1$  provokes an overflow in the upper half of  $U$  and is therefore packed into the second shelf of the Ceiling Packing; see Fig. 11(g). In this case, the total area of squares from  $H_1$  is greater than  $1/4$ . By assumption,  $Q_s$  is placed in  $M_2$  and we get an additional packing area of at least  $7/64$  from small squares; see Property 6. In total, we have  $\|\sigma\| \geq \|Q_1\| + \|\mathcal{P}\| > 1/4 + 7/64 > 11/32$ .

(f)  $Q_1$  overlaps with  $M_1$ :

Because  $Q_1$  intersects  $M_1$ , the lower boundary of  $Q_1$  must have a distance greater than  $3/4$  from the top of  $U$ ; see Fig. 11(h). Thus, the squares of  $H_1$  fill a total area of more than  $3/16 + 2/16 = 10/32$ . As no  $H_1$ -square ever touches the left half of  $\mathcal{H}_\ell$ , we must have an area of at least  $1/2 \cdot 1/8 = 2/32$  occupied by small squares. In total we get  $\|\sigma\| \geq \|Q_1\| + \|\mathcal{P}\| > 10/32 + 2/32 > 11/32$ .  $\square$



**Fig. 11** Different types of collision that may appear if the total area of the input exceed  $11/32$ .



**Fig. 12** Different choices in the lower-bound sequence: (a) Packing after choosing a side position. (b) Packing after choosing a center position. (c) The recurring pattern. (d) Packing a last square.

### 3 Packing into a Dynamic Container

Now we discuss the problem of online packing a sequence of squares into a dynamic square container. At each stage, the container must be large enough to accommodate all objects; this requires keeping the container tight early on, but may require increasing its edge length appropriately during the process.

In the following, we give a non-trivial family of instances, which prove that no online algorithm can maintain a packing density greater than  $3/7$  for an arbitrary input sequence of squares and introduce an online square packing algorithm that maintains a packing density of  $1/8$  for an arbitrarily input sequence of squares.

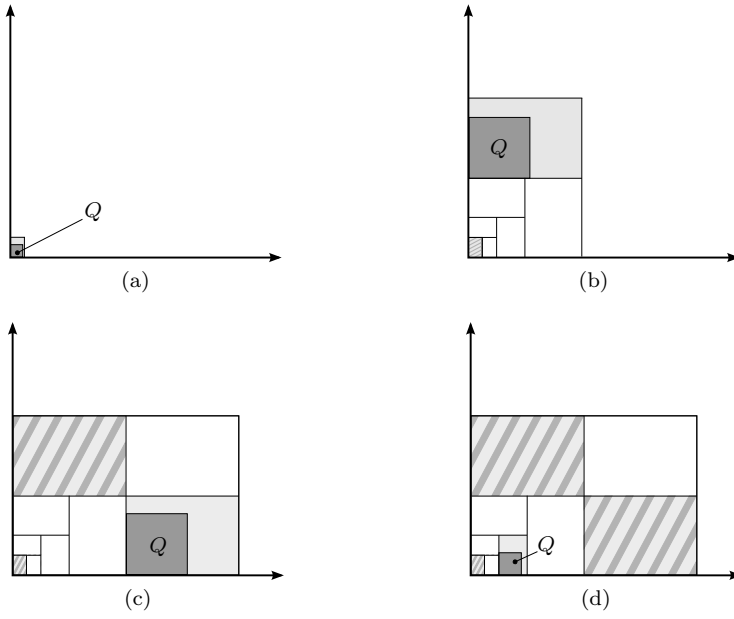
#### 3.1 An Upper Bound on $\delta$

If the total area of the given sequence is unknown in advance, the problem of finding a dense online packing becomes harder. As it turns out, a density of  $1/2$  can no longer be achieved.

**Theorem 4** *There are sequences for which no deterministic online packing algorithm can maintain a density strictly greater than  $3/7 \approx 0.4286$ .*

*Proof.* We construct an appropriate sequence of squares, depending on what choices a deterministic player makes; see Fig. 12. At each stage, the player must place a square  $Q_3$  into a corner position (Fig. 12(a)) or into a center position (Fig. 12(b)); the opponent responds by either requesting another square of the same size (a), or two of the size of the current spanning box. This is repeated. A straightforward computation shows that the asymptotic density becomes  $2/3$ , if the player keeps choosing side positions, and  $3/4$ , if he keeps choosing center positions; for mixed choices, the density lies in between. Once the density is arbitrarily close to  $3/4$ , with the center position occupied, the opponent can request a final square of size  $3/4$  of the current spanning box, for a density close to  $\frac{3/4 + (3/4)^2}{(7/4)^2} = 3/7$ ; if the center position does not get occupied, the density is even worse.  $\square$





**Fig. 13** The modified Brick-Packing algorithm for an input square  $Q$ . Occupied bricks are hatched, free bricks are blank. (a) A first square gets placed into the lower left corner,  $B_{max} = S(Q)$ . (b) If  $S(Q) > B_{max}$ , we double  $B_{max}$  until  $Q$  fits. (c) If  $Q$  does not fit into  $B_{max}$ , but  $\|S(Q)\| < \|B_{max}\|$ , we double  $B_{max}$  and subdivide the resulting brick. (d) If  $Q$  fits into  $B_{max}$ , we pack it into the smallest free fitting subbrick.

### 3.2 A Lower Bound on $\delta$

When placing squares into a dynamic container, we cannot use our Recursive Shelf Algorithm, as it requires allocating shelves from all four container boundaries, which are not known in advance. However, we can adapt the Brick Algorithm by [7]: we consider bricks with side lengths equal to a power of  $\sqrt{2}$  (and aspect ratio  $1/\sqrt{2}$  or  $\sqrt{2}$ ). We let  $B_k$  denote the brick of size  $(\sqrt{2}^k, \sqrt{2}^{k+1})$  and let  $S(Q)$  denote the smallest brick  $B_i$  that may contain a given square  $Q$ .

There are two crucial modifications: (1) The first square  $Q$  is packed into a brick of size  $S(Q)$  with its lower left corner in the origin and (2) instead of always subdividing the existing bricks (starting with three fixed ones), we may repeatedly double the current maximum existing brick  $B_{max}$  to make room for large incoming squares. Apart from that, we keep the same packing scheme: Place each square  $Q$  into (a subbrick of) the smallest free brick that can contain  $Q$ ; see Fig. 13 for an illustration.

**Theorem 5** *For any input sequence of squares, the Dynamic Brick Algorithm maintains a packing density of at least  $1/8$ .*

*Proof.* By construction, every occupied brick has a density of at least  $1/(2\sqrt{2})$ . It is easy to see that in every step of the algorithm at most half the area of  $B_{max}$  consists of free bricks; compare [7]. Because  $B_{max}$  always contains all occupied

bricks (and thus all packed squares), the ratio of  $\|B_{max}\|$  to the area of the smallest enclosing square is at least  $1/\sqrt{2}$ . Therefore, the algorithm maintains an overall density of at least  $(1/(2\sqrt{2})) \cdot (1/2) \cdot (1/\sqrt{2}) = 1/8$ .  $\square$

### 3.3 Minimizing Container Size

The above results consider the worst-case ratio for the packing density. A closely related question is the online optimization problem of maintaining a square container with minimum edge length. The following is an easy consequence of Theorem 5, as a square of edge length  $2\sqrt{2}$  can accommodate a unit area when packed with density  $1/8$ . By considering optimal offline packings for the class of examples constructed in Theorem 4, it is straightforward to get a lower bound of  $4/3$  for any deterministic online algorithm.

**Corollary 2** *Dynamic Brick Packing provides a competitive factor of  $2\sqrt{2} = 2.82\dots$  for packing an online sequence of squares into a square container with small edge length. The same problem has a lower bound of  $4/3$  for the competitive factor.*

## 4 Conclusion

We have presented progress on two natural variants of packing squares into a square in an online fashion. The most immediate open question remains the critical packing density for a fixed container, where the correct value may actually be less than  $1/2$ . Online packing into a dynamic container remains wide open. There is still slack in both bounds, but probably more in the lower bound.

There are many interesting related questions. What is the critical density (offline and online) for packing circles into a unit square? This was raised by Demaine et al. [1]. In an offline setting, there is a lower bound of  $\pi/8 = 0.392\dots$ , and an upper bound of  $\frac{2\pi}{(2+\sqrt{2})^2} = 0.539\dots$ , which is conjectured to be tight. Another question is to consider the critical density as a function of the size of the largest object. In an offline context, the proof by Moon and Moser provides an answer, but little is known in an online setting.

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